

A modified Conditional Least Squares estimator for parameters in a class of bilinear models

Uno stimatore modificato dei minimi quadrati condizionati dei parametri per una classe di modelli bilineari

Francesco Giordano

Dipartimento di Scienze Economiche e Statistiche, Università di Salerno
e-mail: giordano@unisa.it

Riassunto: Questo lavoro si pone in una classe di modelli bilineari per derivare un nuovo stimatore dei parametri basato sui minimi quadrati condizionati. Grahn (1995) ha presentato uno stimatore dei minimi quadrati condizionati per diverse classi di modelli bilineari. Si dimostra che il nuovo stimatore gode delle stesse proprietà dimostrate da Grahn (1995) sotto condizioni meno restrittive. Inoltre, si pongono in evidenza anche alcuni vantaggi computazionali.

Keywords: bilinear models, CLS estimator, central limit theorem, mixed moments.

1. Introduction

Bilinear models have two important fields of application: the first is related to their combined use with other models such as SETAR, GARCH; the second deals with the fitting of real time series observations by a proper bilinear model. For example the sharp spikes, which are typical features in river data sets, are well explained by this model. In both cases a common problem attains to the parameters estimation that Grahn (1995) faced by introducing the Conditional Least Squares (CLS) estimator for parameters of different classes of bilinear models.

Let X_t be a real-valued stochastic process which satisfies the following difference equations:

$$X_t = \sum_{i=1}^p a_i X_{t-i} + \sum_{i=1}^q c_i \varepsilon_{t-i} + \varepsilon_{t-k} \sum_{j=1}^{r-w+1} b_j X_{t-w+1-j} + \varepsilon_t \quad (1)$$

where ε_t is an i.i.d. zero-mean white noise process with variance $\sigma^2 > 0$, a_i , c_i and $b_j \in \mathcal{R}$ are unknown parameters and $w = \max(k, q) + 1$.

The process X_t in (1) is a subclass of type (I) in Grahn (1995), therefore the parameters a_i and c_i can be estimated by following the same procedure there shown. Instead, in this paper we propose a new CLS estimator for parameters b_j and σ^2 in (1).

We show that the new CLS estimator has the same properties as the classical one but it is based on less restrictive conditions. Besides, we introduce some computational tools to derive the new CLS estimator.

Some definitions and assumptions are reported in section 2. The main results about the new CLS estimator are presented in section 3. A simulation experiment and some concluding remarks are illustrated in section 4.

2. Definitions and assumptions

The process X_t in (1) can be written in a suitable state-space form as in Liu and Brockwell (1988). They showed that, under some conditions on parameters, there exists a causal, ergodic and strictly stationary solution for (1) (Liu and Brockwell, 1988) and such a solution holds with respect to the norm of the space L^2 . Using the same technique as in Liu and Brockwell (1988), and by theorem 3.1 in Grahn (1995), we can derive the conditions under which the solution to (1) holds with respect to the norm of L^m .

So we can state the following assumptions:

A1) If $E(|\varepsilon_t|^3) < \infty$ and $\rho(\Gamma_3) < 1$ then there exists a solution to (1) in the sense of L^3 norm. Γ_3 is a matrix defined by parameters in (1) which can be built following either Liu and Brockwell (1988) or Grahn (1995). $\rho(A)$ is the maximum eigenvalue of the matrix A .

A2) the same given in Grahn (1995), which is related to the AR part of (1).

A3) the same given in Grahn (1995), which concerns the MA part of (1).

Finally, we have to define the CA condition between a parameter θ and its CLS estimator $\hat{\theta}_{CLS}$ (Grahn, 1995).

In particular, we state that $(\theta, \hat{\theta}_{CLS})$ fulfills the CA condition of order m if

- i) there exists both a finite dimensional vector M_m whose components belong to the set of the theoretical mixed moments of the process X_t in (1) up to order m and a function $f \in C^1[I(M_m)]$ such that $\theta = f(M_m)$ ($C^1[I(M_m)]$ defines the space of continuously differentiable function in a suitable neighborhood $I(M_m)$ of M_m).
- ii) $\left| \hat{\theta}_{CLS} - f(\hat{M}_m) \right| = o_p(N^{-1/2})$, where $\hat{M}_m$ denotes the vector of the corresponding empirical estimates of the theoretical moments M_m and N is the length of the observed time series.

The condition (A1) implies that $E(|X_t|^3) < \infty$ and X_t in (1) is strictly stationary.

3. Main results

The model (1) contains parameters in the linear part, a_i and c_i , and parameters b_j in the bilinear part. Without loss of generality we can consider only the bilinear part because a_i and c_i can be estimated according to the procedure of Grahn (1995). Therefore we have

$$X_t = \varepsilon_{t-k} \sum_{j=1}^{r-w+1} b_j X_{t-w+1-j} + \varepsilon_t \quad (2)$$

The proposed CLS estimator is defined by the linear equation

$$E(X_t X_{t-w+1} | \varepsilon_{t-w}, \varepsilon_{t-w-1}, \dots) = \sum_{j=1}^{r-w+1} b_j \sigma^2 X_{t-w+1-j}.$$

So

$$\sum_t X_t X_{t-w+1} = \sum_{j=1}^{r-w+1} b_j \sigma^2 \sum_t X_{t-w+1-j}. \quad (3)$$

Fix $\beta_j \equiv b_j \sigma^2$. We get $\hat{\beta}_j$ by applying the ordinary least squares to (3). $\hat{\beta}_j$ is spanned only by $\{X_{t-w+1-j}\}$ with $1 \leq j \leq r-w+1$. So, $\hat{\beta}_j$ is defined up to the third mixed moments of X_t in (2).

The classical CLS estimator considers more than one linear equation. In particular we have w different linear equations with $w \geq 2$. Now, we can prove the following lemma

Lemma 1 *If (A1) holds in (2) then $\hat{\beta}_j \xrightarrow{a.s.} \beta_j$. Moreover, if $\rho(\Gamma_6) < 1$ and $E(\varepsilon_t^6) < \infty$ then $(\hat{\beta}_j, \beta_j)$ fulfills CA condition of order 3.*

Now put $K_j \equiv \hat{\beta}_j$, where K_j can be also written as $K_j = \hat{b}_j \bar{X}_2 \left(1 - \sum_{j=1}^{r-w+1} \hat{b}_j K_j\right)$ with $1 \leq j \leq r-w+1$ and $\bar{X}_2 = 1/N \sum_t X_t^2$.

$\hat{b}_j$ is the new CLS estimator which depends on K_j and $\bar{X}_2$ only.

There exists an s such that $K_s \neq 0$, with $s \in \{1, \dots, r-w+1\}$. So, it is easy to derive $\hat{b}_j = K_j / K_s \hat{b}_s$, $\forall j \neq s$ and

$$\hat{b}_s^2 \left(\sum_{j=1}^{r-w+1} K_j^2 \right) - K_s \hat{b}_s + K_s^2 / \bar{X}_2 = 0 \quad (4)$$

Let ${}_1\hat{b}_s$ be a solution of (4) and denote with ${}_1\hat{b}_j$ the other corresponding estimators $\forall j \neq s$.

The new CLS estimator of σ^2 is ${}_1\hat{\sigma}^2 = \bar{X}_2 \left(1 - \sum_{j=1}^{r-w+1} {}_1\hat{b}_j K_j\right)$.

We can prove the following two lemmas:

Lemma 2 *Given the condition of lemma 1, ${}_1\hat{b}_s$ is asymptotically real a. s.*

Before we state the next lemma, put $g^2 = \sum_{j=1}^{r-w+1} \hat{b}_j^2 \sigma^2$.

Lemma 3 *Given the condition of lemma 1 and $g^2 < 1/2$ then ${}_1\hat{b}_s$ is the unique asymptotically feasible solution to (4) a. s. in the sense that ${}_1\hat{b}_s \xrightarrow{a.s.} b_s$. Moreover, such a solution is with sign minus if $b_s > 0$ and positive if $b_s < 0$.*

If $K_s > 0$ then $b_s > 0$ asymptotically a. s. and similarly for $K_s < 0$, then ${}_1\hat{b}_s$ is the unique solution to (4) (lemma 3) with the sign *minus* if $K_s > 0$ and with sign *plus* if $K_s < 0$. So we have the new CLS estimators ${}_1\hat{b}_j$, $\forall j$, and ${}_1\hat{\sigma}^2$.

The new CLS estimators are now completely defined and we can state a theorem for their properties.

Theorem 1 *If the condition of lemma 1 holds and $g^2 < 1/2$ then ${}_1\hat{b}_j \xrightarrow{a.s.} b_j \forall j$. Moreover, if $\rho(\Gamma_6) < 1$ and $E(\varepsilon_t^6) < \infty$ then*

$$\sqrt{N} \left({}_1\hat{b}_j - b_j \right) \xrightarrow{d} N(0, V^2)$$

where V^2 is the variance of the Normal limit distribution.

The same results hold for ${}_1\hat{\sigma}^2$.

With slight modifications to the new CLS estimators we can apply them to the full model (1) as well. In particular, we have to consider some minor changes to linear equations in (3), due to the MA part.

4. Simulations and concluding remarks

We present a Monte-Carlo simulation to evaluate the performance of the new CLS estimators. We consider a simple model

$$X_t = \varepsilon_t + b\varepsilon_{t-1}X_{t-2} \quad (5)$$

where $\varepsilon_t \sim i.i.d. \quad N(0, \sigma^2)$.

The Monte-Carlo runs are 2000 and $\sigma^2 = 1$. Sample sizes are $N = \{200, 500, 1000\}$ and $b = \{0.3, 0.6\}$ which satisfies the conditions in theorem 1. We compute, for each simulated time series, the new CLS estimator ${}_1\hat{b}$ and the classical one $\hat{b}$. Fix ${}_1S_b = \sqrt{N}({}_1\hat{b} - b)$ and $S_b = \sqrt{N}(\hat{b} - b)$.

Table 1: Estimators ${}_1\hat{b}$ and $\hat{b}$. $M = \text{Mean}$, $Sd = \text{Standard deviation}$.

	${}_1S_b \quad (b = 0.3)$		$S_b \quad (b = 0.3)$		${}_1S_b \quad (b = 0.6)$		$S_b \quad (b = 0.6)$	
N	M	Sd	M	Sd	M	Sd	M	Sd
200	-0.01	1.90	-0.16	1.57	-1.75	2.31	-1.23	2.74
500	-0.02	1.76	-0.11	1.56	-1.56	3.29	-0.88	3.83
1000	-0.02	1.70	-0.07	1.61	-1.55	4.25	-1.00	4.31

Looking at Table 1 we can note that both ${}_1\hat{b}$ and $\hat{b}$ underestimate the true parameter as pointed out in Grahn (1995). When b grows there is a reduced variance effect for ${}_1\hat{b}$.

The new proposed CLS estimator can be called "minimal" because it uses only the "minimal" mixed moments to have a consistent estimator of the parameters. In fact, ${}_1\hat{b}$ uses the mixed moments of X_t up to order 3, while $\hat{b}$ up to order 4 (Grahn, 1995). Besides, the proposed CLS estimator needs less restrictive conditions on moments for both strong consistency and weak convergence. The technical condition in lemma 3 and theorem 1, $g^2 < 1/2$, can be relaxed.

Finally, we have a computationally advantage because the new CLS estimator is derived using only one linear equation as shown in section 3. For example, we can avoid some numerical problems related to quasi non-invertible matrix when we use a bootstrap method to estimate the unknown parameters.

References

- Grahn T. (1995) A Conditional Least Squares approach to bilinear time series estimation, *Journal of Time Series Analysis*, 16, 509–529.
- Granger C. W., Andersen A. P. (1978) *An introduction to bilinear time series models*, Vandenhoeck and Ruprecht, Göttingen.
- Hall P., Heyde C. C. (1980) *Martingale limit theory and its Application*, Academic Press, New York.
- Liu J., Brockwell P. J. (1988) On the general bilinear time series model *J. Appl. Prob.*, 25, 553–564
- Tong H. (1990) *Non-linear time series*, Oxford University Press, Oxford.